

Sine distance for quantum states

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We thoroughly analyse the distance between quantum states that has been applied to state-dependent cloning and partly studied in the previous work of the author [Phys. Rev. A **66**, 042304 (2002)]. Elementary proofs of its significant properties are given.

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I. INTRODUCTION

Over the last twenty years there have been impressive theoretical and experimental advances in actions on single quantum and use of them for information processing. Above all, quantum cryptography [1], quantum factoring [2] and quantum searching [3] are most inspiring. Quantum devices can apparently provide more powerful tools of communication and computation than classical ones. A design of efficient algorithm for some quantum information processing task demands that we compare results of tested quantum operations. So mathematical techniques of quantum information theory must include quantitative measures of closeness of quantum states.

It would seem that for pure states the square-overlap provides such a measure. But quantum information is a growing branch with many facets. Natural as the square-overlap is, it is not able to give the best measure in all respects. In Ref. [4] the author offered a new approach to the state-dependent cloning. As it is well-known [5], the majority of the studies of cloning uses the figures of merit based on the fidelity. For pure states it is reduced to the square-overlap. The new figure of merit, called "relative error", is based on *sine of angle between two states* [4]. As it turned out, this figure of merit is dualistic to those arising from the square-overlap. In Ref. [6] the above approach was extended to mixed-state cloning. So the study of the relative error has allowed us to complement the portrait of state-dependent cloning [4,6]. Thus, there may be more than one way to fit the problem of state closeness, even if the states are pure.

We see that the problem of building a good distance for quantum states is inevitable. Moreover, we rather need some set of reliable measures, complementing each other. In fact, it is impossible to foresee all potential questions, because many fields are still undeveloped. Here several example may be pointed out. Since many results in the quantum cloning used the fidelity, the writers of Ref. [5] have stated such a question. What about other figures of merit of clones? Reviewing the ideas of quantum information within the frames of relativity theory, Peres

and Terno raised a few important open problems [7]. Of course, any workable notion of distance must be bound in the natural way. First, a distance should have clear physical meaning. Second, it should have a direct expression in the terms of density matrices. At last, there is specific "distance" property: it must be a metric.

The aim of the present work is to further clarify understanding of the sine of angle between two states as a distance measure. It will be referred as to "sine distance". As it is shown in Refs. [4,6], the use of the sine distance as distance measure provides new and fruitful viewpoint on the state-dependent cloning. More recently, the writers of Ref. [8] found that the above distance naturally arises in the context of quantum computation. But the sine distance is not previously studied as independent notion. The contribution of the paper is to fill up this lacuna. To simplify the exposition, we separately consider the case of pure states. It is such a case that is natural. First, many key ideas of quantum information were discovered in the models those deal only with pure states. Second, the analysis of transformations of mixed states demands more powerful techniques. At last, when the states are impure, a qualitative restatement of considered task may be needed. For example, for pure states the cloning is equivalent of broadcasting, but for mixed states the cloning is very special case of broadcasting [9]. Starting with pure-state case, we then examine the general case of mixed states. After briefly reviewing of background material on impure states, we establish basic properties of the sine distance. Further, the sine distance is considered in the context of quantum operations. The principal role of this is to provide the clear justification why the notion of sine distance is reliable and convenient. In the present work by the quantum states we mean that ones all are normalized.

II. THE CASE OF PURE STATES

In this section we shall give a clear definition of "sine distance" for pure states. As every, we define the angle $\delta(x, y) \in [0; \pi/2]$ between pure states $|x\rangle$ and $|y\rangle$ by

$$\delta(x, y) := \arccos |\langle x|y \rangle|. \quad (2.1)$$

We will also write δ_{xy} in bulky expressions. States $|x\rangle$ and $|y\rangle$ are identical if and only if $\delta_{xy} = 0$.

Definition 1 *Sine distance between pure states $|x\rangle$ and $|y\rangle$ is defined by*

$$d(x, y) := \sin \delta(x, y). \quad (2.2)$$

The pure state is a ray in the Hilbert space. So, for given two states $|x\rangle$ and $|y\rangle$ we can always suppose that $\langle x|y\rangle$ is a nonnegative real number. Let us denote

$$|x\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle, \quad (2.3)$$

$$|y\rangle = \sin\theta|0\rangle + \cos\theta|1\rangle, \quad (2.4)$$

that is customary in state-dependent cloning [10]. The vectors $|0\rangle$ and $|1\rangle$ are orthonormal and $2\theta \in [0; \pi/2]$. So, the overlap between $|x\rangle$ and $|y\rangle$ is $|\langle x|y\rangle| = \sin 2\theta$, and by Eqs. (2.1) and (2.2) we therefore have

$$d(x, y) = \cos 2\theta. \quad (2.5)$$

The convenience of parametrization by Eqs. (2.3) and (2.4) is that for each linear operator L we have

$$\langle x|L|x\rangle - \langle y|L|y\rangle = \{\langle 0|L|0\rangle - \langle 1|L|1\rangle\} d(x, y). \quad (2.6)$$

It turns out that for any quantum operation the quantity $d(x, y)$ estimates the difference between probabilities of processes beginning with inputs $|x\rangle$ and $|y\rangle$ respectively. Recall that a quantum operation \mathcal{E} is formal description of physical process which starts with an input state σ of quantum system S and results in an output state

$$\sigma' := \frac{\mathcal{E}(\sigma)}{\text{tr}\{\mathcal{E}(\sigma)\}}$$

of (generally another) quantum system S' . The normalizing divisor is trace over the Hilbert space \mathcal{H}' of S' and gives the probability that such a process occurs. So we have $0 \leq \text{tr}\{\mathcal{E}(\sigma)\} \leq 1$. The domain of \mathcal{E} is real vector space of Hermitian operators on the input space \mathcal{H} . The range of \mathcal{E} is a subset of real vector space of Hermitian operators on the output space \mathcal{H}' . It is necessary that this map be linear and completely positive [11].

The operator-sum representation is a key result of the theory of quantum operations. That is [11], the map \mathcal{E} is a quantum operation if and only if

$$\mathcal{E}(\sigma) = \sum_{\mu} E_{\mu} \sigma E_{\mu}^{\dagger}$$

for some set of operators $\{E_{\mu}\}$. These operators map the input space \mathcal{H} to the output space \mathcal{H}' and satisfy

$$\sum_{\mu} E_{\mu}^{\dagger} E_{\mu} \leq \mathbf{1}. \quad (2.7)$$

It is necessary for proper probabilistic treatment. The set $\{E_{\mu}\}$ completely specifies a quantum operation. For given set of such operators let \mathcal{E}_{ν} denotes a operation specified by single operator E_{ν} , i.e. $\mathcal{E}_{\nu}(\sigma) := E_{\nu} \sigma E_{\nu}^{\dagger}$. According to the terminology of Ref. [12], such a quantum operation is contained in the class of *ideal* operations. The following statement is a basic result of this section.

Proposition 1 *If the set $\{E_{\mu}\}$ of operators specifies a quantum operation \mathcal{E} then*

$$|\text{tr}\{\mathcal{E}(|x\rangle\langle x|)\} - \text{tr}\{\mathcal{E}(|y\rangle\langle y|)\}| \leq d(x, y), \quad (2.8)$$

$$\sum_{\mu} |\text{tr}\{\mathcal{E}_{\mu}(|x\rangle\langle x|)\} - \text{tr}\{\mathcal{E}_{\mu}(|y\rangle\langle y|)\}| \leq 2d(x, y). \quad (2.9)$$

Proof Because the trace of sum of operators is equal to the sum of traces of these operators, we obtain

$$\text{tr}\{\mathcal{E}(|v\rangle\langle v|)\} = \sum_{\mu} \text{tr}\{\mathcal{E}_{\mu}(|v\rangle\langle v|)\} \quad (2.10)$$

for $v = x, y$. Due to $\text{tr}\{|u'\rangle\langle v'|\} = \langle v'|u'\rangle$, each term of the latter sum can be expressed as

$$\text{tr}\{E_{\mu}|v\rangle\langle v|E_{\mu}^{\dagger}\} = \langle v|\mathbf{T}_{\mu}|v\rangle, \quad (2.11)$$

where $\mathbf{T}_{\mu} := E_{\mu}^{\dagger} E_{\mu}$. By Eqs. (2.10) and (2.11) we have

$$\text{tr}\{\mathcal{E}(|v\rangle\langle v|)\} = \langle v|\mathbf{T}|v\rangle, \quad (2.12)$$

where positive operator $\mathbf{T} := \sum_{\mu} \mathbf{T}_{\mu}$. By Eqs. (2.12) and (2.6), we rewrite the left-hand side of Eq. (2.8) as

$$|\langle 0|\mathbf{T}|0\rangle - \langle 1|\mathbf{T}|1\rangle| d(x, y), \quad (2.13)$$

By Eq. (2.7), we get $\mathbf{0} \leq \mathbf{T} \leq \mathbf{1}$, $0 \leq \langle u|\mathbf{T}|u\rangle \leq 1$ and

$$-1 \leq \langle 0|\mathbf{T}|0\rangle - \langle 1|\mathbf{T}|1\rangle \leq +1.$$

So due to expression (2.13) we obtain Eq. (2.8).

Continuing, due to Eqs. (2.11) and (2.6) we can write

$$\begin{aligned} & |\text{tr}\{\mathcal{E}_{\mu}(|x\rangle\langle x|)\} - \text{tr}\{\mathcal{E}_{\mu}(|y\rangle\langle y|)\}| \\ &= |\langle 0|\mathbf{T}_{\mu}|0\rangle - \langle 1|\mathbf{T}_{\mu}|1\rangle| d(x, y) \\ &\leq \{\langle 0|\mathbf{T}_{\mu}|0\rangle + \langle 1|\mathbf{T}_{\mu}|1\rangle\} d(x, y). \end{aligned}$$

To sum over all μ 's, we see that the left-hand side of Eq. (2.9) is not larger than $\{\langle 0|\mathbf{T}|0\rangle + \langle 1|\mathbf{T}|1\rangle\} d(x, y)$. By $\mathbf{0} \leq \mathbf{T} \leq \mathbf{1}$, the latter does not exceed $2d(x, y)$. ■

It must be stressed that the general upper bounds, given by Proposition 1, are least and cannot be refinement. We shall now show this fact. In the remainder of this section let \mathbf{P} be an operator such that $\mathbf{0} \leq \mathbf{P} \leq \mathbf{1}$ and $\text{span}\{|x\rangle, |y\rangle\}$ is a subspace of its kernel.

Proposition 2 *If a quantum operation \mathcal{E} reaches the first upper bound of Proposition 1 for given states $|x\rangle$ and $|y\rangle$ then either $\mathbf{T} = |0\rangle\langle 0| + \mathbf{P}$ or $\mathbf{T} = |1\rangle\langle 1| + \mathbf{P}$.*

Proof Take a basis $\{|j\rangle\}$ containing kets $|0\rangle$ and $|1\rangle$ from the parametrization by Eqs. (2.3) and (2.4). As the proof of Proposition 1 shows, the left-hand side of Eq. (2.8) is equal to $|c_{00} - c_{11}| d(x, y)$, where $c_{jk} := \langle j|\mathbf{T}|k\rangle$. So the equality in Eq. (2.8) holds if and only if

$$|c_{00} - c_{11}| = 1. \quad (2.14)$$

Due to $\mathbf{0} \leq \mathbf{T} \leq \mathbf{1}$ we have $0 \leq c_{jj} \leq 1$ for all values of label j . Under the latter, Eq. (2.14) is satisfied in two cases: (i) $c_{00} = 1$ and $c_{11} = 0$; (ii) $c_{00} = 0$ and $c_{11} = 1$. In the case (i) we obtain

$$\mathbf{T} = |0\rangle\langle 0| + c_{01}|0\rangle\langle 1| + c_{10}|1\rangle\langle 0| + \mathbf{P}, \quad (2.15)$$

where operator

$$\mathbf{P} := \sum_{j,k=2}^{N-1} c_{jk}|j\rangle\langle k| \quad (2.16)$$

and $N := \dim(\mathcal{H})$. The action of operator \mathbf{T} in subspace $\text{span}\{|x\rangle, |y\rangle\}$ is described by the matrix

$$\begin{pmatrix} 1 & \alpha - i\beta \\ \alpha + i\beta & 0 \end{pmatrix}. \quad (2.17)$$

Here α and β are real, and $c_{01}^* = \alpha + i\beta = c_{10}$ by Hermiticity. Due to positivity, both eigenvalues of the matrix (2.17) are nonnegative that takes place if and only if $\alpha = \beta = 0$. So $c_{01} = c_{10} = 0$ and from Eq. (2.15) we obtain $\mathbf{T} = |0\rangle\langle 0| + \mathbf{P}$. By a parallel argument, in the case (ii) we get $\mathbf{T} = |1\rangle\langle 1| + \mathbf{P}$. To satisfy condition $\mathbf{0} \leq \mathbf{T} \leq \mathbf{1}$, the operator \mathbf{P} must obey $\mathbf{0} \leq \mathbf{P} \leq \mathbf{1}$. Due to definition by Eq. (2.16), $\text{span}\{|x\rangle, |y\rangle\}$ is a subspace of kernel of operator \mathbf{P} . ■

Just as the bound given by Eq. (2.8), the upper bound given by Eq. (2.9) is also attainable. For example, the equality in Eq. (2.9) is reached by the quantum operation

$$\mathcal{E}(|v\rangle\langle v|) = E_0|v\rangle\langle v|E_0^\dagger + E_1|v\rangle\langle v|E_1^\dagger,$$

where two mapping operators satisfy $E_0^\dagger E_0 = |0\rangle\langle 0|$ and $E_1^\dagger E_1 = |1\rangle\langle 1|$. This follows from Proposition 2. Since both bounds of Proposition 1 deal with probabilities that corresponding process occurs, it provides well-motivated physical meaning of the sine distance.

However, the real devices are inevitably exposed to noise. Key result of a quantum system interacting with its environment is the loss of superposition, called "decoherence" [13]. It is its action that is to undo the interference of states used in data processing, replacing them instead with mixtures of states. The solution of subroutine problem required a consideration of quantum circuits with mixed states [14]. As it is shown in Ref. [15], the cloning machine, which can input any mixed state in symmetric subspace, is necessary in quantum information. So, however hard the careful examination of mixed states may be from the technique viewpoint, we are to develop it. We shall now extend the notion of sine distance to the case of mixed states.

III. THE CASE OF MIXED STATES

As it is well known, the square-overlap $|\langle x|y\rangle|^2$ is the probability that $|y\rangle$ passes the yes/no test of "being the state $|x\rangle$." This clear physical meaning gives us a better understanding of why the square-overlap ensures a natural way to distinguish pure states. However, there is no evident analog of yes/no test for mixed states. Nevertheless, we can extend to mixed states a few notions which are useful in the case of pure states. This is provided by the concept of *purifications*.

According to the "decoherence" viewpoint [13], any mixed state is describing the reduced states of a subsystem S entangled with the environment. The total system is being in a pure state. If the quantum system S is considered then we append system Q , which is a copy of S .

Widening the above viewpoint, we can imagine that a mixed state σ of S arises by partial trace operation from pure state of extended system SQ . Namely, there is a pure state $|X\rangle$, called "purification", for which [16]

$$\sigma = \text{tr}_Q\{|X\rangle\langle X|\}. \quad (3.1)$$

For any mixed states its purification can be made, and for given one such a pure state is not unique [16].

In Ref. [17] we have defined the angle $\Delta(\sigma, \rho) \in [0; \pi/2]$ between mixed states σ and ρ by

$$\Delta(\sigma, \rho) := \min \delta(X, Y), \quad (3.2)$$

where the minimum is taken over all purifications $|X\rangle$ of σ and $|Y\rangle$ of ρ . The properties of angle between mixed states are listed in Ref. [17]. In particular, we have

$$\Delta(\sigma, \rho) \leq \Delta(\sigma, \omega) + \Delta(\rho, \omega). \quad (3.3)$$

We are now able to extend the notion of "sine distance" to the case of mixed states.

Definition 2 *Sine distance between mixed states σ and ρ is defined by*

$$d(\sigma, \rho) := \sin \Delta(\sigma, \rho). \quad (3.4)$$

The sine distance can simply be expressed in terms of fidelity function. Recall that the fidelity function generalizes the square-overlap. More precisely, for given mixed states σ and ρ of system S the fidelity is defined as

$$F(\sigma, \rho) := \max |\langle X|Y\rangle|^2, \quad (3.5)$$

where the maximum is taken over all purifications $|X\rangle$ of σ and $|Y\rangle$ of ρ [16]. Using Eqs. (3.2), (3.4) and (3.5), it is easy to verify that

$$d(\sigma, \rho) = \sqrt{1 - F(\sigma, \rho)}. \quad (3.6)$$

The definition by Eq. (3.5) gives a kind of physical meaning of the fidelity and makes many of its properties to be clear. But this formula does not provide a calculational tool for evaluating the fidelity function. Fortunately, the notion of fidelity is equivalent to Uhlmann's "transition probability" [16]. By Uhlmann's "transition probability" formula [18], the fidelity of states σ and ρ is

$$F(\sigma, \rho) = \left\{ \text{tr}_S \left[(\sqrt{\sigma} \rho \sqrt{\sigma})^{1/2} \right] \right\}^2. \quad (3.7)$$

So, the fidelity and, therefore, the sine distance can directly be expressed in the terms of density operators.

The basic properties of the sine distance can be obtained from the definition by Eq. (3.4) and features of the fidelity function. In particular, the sine distance is a metric. These properties are stated by the following.

Theorem 1 *Sine distance ranges between 0 and 1, and $d(\sigma, \rho) = 0$ if and only if $\sigma = \rho$. It is symmetric, i.e. $d(\sigma, \rho) = d(\rho, \sigma)$. It obeys the triangle inequality:*

$$d(\sigma, \rho) \leq d(\sigma, \omega) + d(\rho, \omega). \quad (3.8)$$

Its square is convex: if $q, r \geq 0$ and $q + r = 1$ then

$$d^2(\sigma, q\rho + r\omega) \leq q d^2(\sigma, \rho) + r d^2(\sigma, \omega) . \quad (3.9)$$

Proof The first and second properties are corollaries of Definition 2. So we will prove only the triangle inequality and that the square of sine distance is convex. To establish Eq. (3.8), we consider two cases:

$$0 \leq \Delta(\sigma, \omega) + \Delta(\rho, \omega) \leq \pi/2 ; \quad (3.10)$$

$$\pi/2 \leq \Delta(\sigma, \omega) + \Delta(\rho, \omega) \leq \pi . \quad (3.11)$$

By definition, the angle lies in the range $[0; \pi/2]$, where the sine is a nondecreasing function. Due to Eq. (3.3), in the case of Eq. (3.10) we obtain

$$\begin{aligned} \sin \Delta_{\sigma\rho} &\leq \sin \Delta_{\sigma\omega} \cos \Delta_{\rho\omega} + \cos \Delta_{\sigma\omega} \sin \Delta_{\rho\omega} \\ &\leq \sin \Delta_{\sigma\omega} + \sin \Delta_{\rho\omega} . \end{aligned}$$

Thus, in the case Eq. (3.10) the triangle inequality for sine distance is corollary of the one for angle. But in the case Eq. (3.11) it is not so! Here an independent proof is wanted. This need is met by Lemma of Appendix A. To prove Eq. (3.9), we shall use the concavity of fidelity. That is [16], for $q, r \geq 0$ and $q + r = 1$ there holds

$$F(\sigma, q\rho + r\omega) \geq qF(\sigma, \rho) + rF(\sigma, \omega) . \quad (3.12)$$

Due to Eq. (3.6) the latter can be rewritten as

$$1 - d^2(\sigma, q\rho + r\omega) \geq q \{1 - d^2(\sigma, \rho)\} + r \{1 - d^2(\sigma, \omega)\} .$$

By $q + r = 1$, the latter provides Eq. (3.9). ■

It is not incurious that in some cases the sine distance shows concavity. Namely, for each $\sigma = |x\rangle\langle x|$ we have

$$d(\sigma, q\rho + r\omega) \geq qd(\sigma, \rho) + rd(\sigma, \omega) . \quad (3.13)$$

Indeed, the fidelity function of states $|x\rangle\langle x|$ and ρ is equal to $\langle x|\rho|x\rangle$ [16], whence

$$F(\sigma, q\rho + r\omega) = qF(\sigma, \rho) + rF(\sigma, \omega) . \quad (3.14)$$

Due to the Jensen's inequality for concave function,

$$\sqrt{1 - (q\zeta + r\xi)} \geq q\sqrt{1 - \zeta} + r\sqrt{1 - \xi} .$$

Substituting $\zeta = F(\sigma, \rho)$ and $\xi = F(\sigma, \omega)$ to the latter inequality, by Eqs. (3.6) and (3.14) we obtain Eq. (3.13). Note that for correctness of the above argument the equality in Eq. (3.12) is necessary.

Thus, the sine distance has useful properties. It ranges between 0 and 1, it is a metric on quantum states, and its square is convex. We shall now consider the sine distance within the frames of quantum operations.

IV. ON THE QUANTUM OPERATIONS

We shall now extend the main result of Sect. II to the case of mixed states. The concept of purification provides a direct way to do this.

Theorem 2 If the set $\{E_\mu\}$ of operators specifies a quantum operation \mathcal{E} then

$$|\text{tr}\{\mathcal{E}(\sigma)\} - \text{tr}\{\mathcal{E}(\rho)\}| \leq d(\sigma, \rho) , \quad (4.1)$$

$$\sum_\mu |\text{tr}\{\mathcal{E}_\mu(\sigma)\} - \text{tr}\{\mathcal{E}_\mu(\rho)\}| \leq 2d(\sigma, \rho) . \quad (4.2)$$

Proof Let us define new operators

$$G_\mu := E_\mu \otimes \mathbf{1}_Q , \quad (4.3)$$

those map the space $\mathcal{H} \otimes \mathcal{H}$ to the space $\mathcal{H}' \otimes \mathcal{H}$. Due to Eq. (2.7), these operators satisfy

$$\sum_\mu G_\mu^\dagger G_\mu \leq \mathbf{1}_{SQ} . \quad (4.4)$$

So the set $\{G_\mu\}$ specifies a quantum operation \mathcal{G} with input space $\mathcal{H} \otimes \mathcal{H}$ and output space $\mathcal{H}' \otimes \mathcal{H}$. Take purifications $|X\rangle$ of σ and $|Y\rangle$ of ρ such that $d(\sigma, \rho) = d(X, Y)$. As it is shown in Appendix B, we then have

$$\text{tr}_{S'}\{\mathcal{E}(\sigma)\} = \text{tr}_{S'Q}\{\mathcal{G}(|X\rangle\langle X|)\} , \quad (4.5)$$

$$\text{tr}_{S'}\{\mathcal{E}(\rho)\} = \text{tr}_{S'Q}\{\mathcal{G}(|Y\rangle\langle Y|)\} .$$

Applying Eq. (2.8) to operation \mathcal{G} , by the last two equalities and $d(\sigma, \rho) = d(X, Y)$ we obtain Eq. (4.1).

According to Appendix B, we also have

$$\text{tr}_{S'}\{\mathcal{E}_\mu(\sigma)\} = \text{tr}_{S'Q}\{\mathcal{G}_\mu(|X\rangle\langle X|)\} , \quad (4.6)$$

$$\text{tr}_{S'}\{\mathcal{E}_\mu(\rho)\} = \text{tr}_{S'Q}\{\mathcal{G}_\mu(|Y\rangle\langle Y|)\} ,$$

where ideal operation \mathcal{G}_ν is specified by single operator G_ν . Applying Eq. (2.9) to all quantum operations \mathcal{G}_μ 's, by a parallel argument we obtain Eq. (4.2). ■

The measurement is an important type of quantum operation. In this case the input and output spaces are the same. As pointed out by Everett [19], a general treatment of all observations by the method of projection operators is impossible. The most general quantum measurement is called a *positive operator valued measure*, or POVM [20]. A POVM with M distinct outcomes is specified by a set of M positive operators \mathbf{A}_μ obeying

$$\sum_{\mu=1}^M \mathbf{A}_\mu = \mathbf{1} . \quad (4.7)$$

Note that the number M of different outcomes is not limited above by the dimensionality N , in contrast to von Neumann measurement. If the system S is prepared in state σ , then the probability of μ 'th outcome is [20]

$$p_\mu(\sigma) := \text{tr}\{\sigma \mathbf{A}_\mu\} . \quad (4.8)$$

With each POVM element \mathbf{A}_ν one can associate an ideal quantum operation \mathcal{A}_ν defined by

$$\mathcal{A}_\nu(\sigma) := \sqrt{\mathbf{A}_\nu} \sigma \sqrt{\mathbf{A}_\nu} .$$

Due to the cyclic property of the trace, we then have

$$p_\nu(\sigma) := \text{tr}\{\mathcal{A}_\nu(\sigma)\} . \quad (4.9)$$

Let us define also an operation $\mathcal{A}(\sigma) := \sum_{\mu} \mathcal{A}_{\mu}(\sigma)$. By Eq. (4.7), this quantum operation is trace-preserving, that is $\text{tr}\{\mathcal{A}(\sigma)\} = 1$. Applying Eq. (4.1) to separately taken operation \mathcal{A}_{μ} and Eq. (4.2) to trace-preserving operation \mathcal{A} , we obtain the following result.

Corollary *For arbitrary POVM there holds*

$$|p_{\mu}(\sigma) - p_{\mu}(\rho)| \leq d(\sigma, \rho), \quad (4.10)$$

$$\sum_{\mu=1}^M |p_{\mu}(\sigma) - p_{\mu}(\rho)| \leq 2d(\sigma, \rho). \quad (4.11)$$

Thus, if the sine distance $d(\sigma, \rho)$ is small then probability distributions generated by states σ and ρ for any measurement are close to each other. Note that special cases of Eq. (4.10) were proven in Refs. [4,6].

The trace-preserving operation also is an important type of quantum operation. Considering the quantum circuits with mixed states, the writers of Ref. [14] showed that a general quantum gate performs trace-preserving, completely positive linear map. So it is a trace-preserving operation. Recall that a quantum operation is trace-preserving when the equality in Eq. (2.7) holds, and so for any state σ we have $\text{tr}\{\mathcal{E}(\sigma)\} = 1$.

As it is known [9], the fidelity function cannot decrease under any trace-preserving quantum operation. Due to Eq. (3.6), the sine distance cannot increase under any trace-preserving operation. That is, if quantum operation \mathcal{E} is trace-preserving then

$$d(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq d(\sigma, \rho). \quad (4.12)$$

When operation is not trace-preserving, the contrary inequality can be valid. The quantum state separation is an evident example of such an operation. In the special case of two inputs, the success outcome of separation leads to decrease of the fidelity of two possible state of the system [21]. So the sine distance will be increased.

As it is shown in Ref. [6], such an inequality holds:

$$|F(\sigma, \omega) - F(\rho, \omega)| \leq d(\sigma, \rho).$$

This, when combined with Eq. (4.12), gives the following. If the operation \mathcal{E} is trace-preserving then for arbitrary $\sigma, \rho \in \mathcal{H}_S$ and $\omega' \in \mathcal{H}'_S$ we have

$$|F(\mathcal{E}(\sigma), \omega') - F(\mathcal{E}(\rho), \omega')| \leq d(\sigma, \rho). \quad (4.13)$$

Thus, if the sine distance $d(\sigma, \rho)$ between inputs σ and ρ is small then the fidelities $F(\mathcal{E}(\sigma), \omega')$ and $F(\mathcal{E}(\rho), \omega')$ are nearly equal to each other. So for any choice of standard ω' the outputs $\mathcal{E}(\sigma)$ and $\mathcal{E}(\rho)$ will be poorly distinguishable. In fact, a natural measure of distinction for mixed states is provided by the fidelity function [16]. It is for this reason that the above interpretation of Eq. (4.13) is to be preferred.

To sum up, we can say that the sine distance between two quantum states provides a reliable measure of their closeness. As the results of this section show, if the value

of $d(\sigma, \rho)$ is small then observable effects caused by states σ and ρ will be close to each other. It should be pointed out that the relations derived here can be useful in various contexts.

V. CONCLUSION

We have examined the sine distance for general quantum states and showed the reasons for its use. This distance measure has good formal properties. Namely, it is a metric on quantum states and ranges between 0 and 1, its square is convex. If the sine distance between two states is known then we can estimate the difference between experimental manifestations of these states. Moreover, this measure cannot increase under any trace-preserving quantum operation. So in a single step of quantum computation the distance between outputs does not exceed the distance between inputs.

In addition to the angle and the sine distance, the Bures metric is also used [22]. As it is well known, this metric is equal to the square root of the quantity $(2 - 2\sqrt{F})$, where F denotes the fidelity. Note that the mentioned metrics all are closely related to each other. Which of these three distances is most preferable? One may scarcely maintain that such a formulation of question is justified. Rather, some distance should be preferred in a first kind of tasks, other distance should be preferred in a second kind of tasks, and so on. For example [23], the use of angles in calculations clarifies the origins of bounds for state-dependent cloning, even if a merit of cloning is measured by the global fidelity.

Nevertheless, the following must be emphasized. The sine distance lies in the interval $[0; 1]$, whereas the angle lies in $[0; \pi/2]$ and the Bures measure lies in $[0; \sqrt{2}]$. As the range of distance values, the interval $[0; 1]$ seems more natural. In addition, the sine distance between two states allows to estimate directly a distinction between their observable effects. So the sine distance is a reliable measure of closeness of quantum states.

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APPENDIX A: LEMMA

Lemma If $\alpha, \beta \in [0; \pi/2]$ and $\pi/2 \leq \alpha + \beta \leq \pi$ then

$$\sin \alpha + \sin \beta \geq 1. \quad (\text{A1})$$

Proof At first, it should be pointed out that

$$\sin \alpha + \sin \beta \geq \sin^2 \alpha + \sin^2 \beta \quad (\text{A2})$$

due to $\alpha, \beta \in [0; \pi/2]$. Applying the usual trigonometry, the right-hand side of Eq. (A2) can be rewritten as

$$\frac{1 - \cos 2\alpha}{2} + \frac{1 - \cos 2\beta}{2} = 1 - \cos(\alpha + \beta) \cos(\alpha - \beta).$$

By conditions $\alpha, \beta \in [0; \pi/2]$ and $\pi/2 \leq \alpha + \beta \leq \pi$, we have $\cos(\alpha + \beta) \cos(\alpha - \beta) \leq 0$ and so Eq. (A1). ■

APPENDIX B: REWRITING TRACES

Let the state σ has the spectral decomposition

$$\sigma = \sum_j \lambda_j |a_j\rangle\langle a_j|.$$

Due to the properties of tracing and Eq. (2.11),

$$\begin{aligned} \text{tr}_{S'} \{ \mathcal{E}_\mu(\sigma) \} &= \sum_j \lambda_j \text{tr}_{S'} \{ \mathcal{E}_\mu(|a_j\rangle\langle a_j|) \} \\ &= \sum_j \lambda_j \langle a_j | \mathbf{T}_\mu | a_j \rangle. \end{aligned} \quad (\text{B1})$$

Applying Eq. (2.10) to the latter relation and summing over all μ 's, we have

$$\text{tr}_{S'} \{ \mathcal{E}(\sigma) \} = \sum_j \lambda_j \langle a_j | \mathbf{T} | a_j \rangle. \quad (\text{B2})$$

In terms of Schmidt polar form [24], any purification $|X\rangle$ of σ can be written as

$$|X\rangle = \sum_j \sqrt{\lambda_j} |a_j\rangle \otimes |f_j\rangle,$$

where kets $|f_j\rangle$ form an orthonormal set in \mathcal{H} . Drawing clear analogy with Eqs. (2.11) and (2.12), we can write

$$\text{tr}_{S'Q} \{ \mathcal{G}_\mu(|X\rangle\langle X|) \} = \langle X | \mathbf{L}_\mu | X \rangle, \quad (\text{B3})$$

$$\text{tr}_{S'Q} \{ \mathcal{G}(|X\rangle\langle X|) \} = \langle X | \mathbf{L} | X \rangle, \quad (\text{B4})$$

where $\mathbf{L}_\mu := G_\mu^\dagger G_\mu$ and $\mathbf{L} := \sum_\mu \mathbf{L}_\mu$ act in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. According to the definition by Eq. (4.3),

$$G_\mu |X\rangle = \sum_j \sqrt{\lambda_j} E_\mu |a_j\rangle \otimes |f_j\rangle.$$

The overlap of this vector with the self is equal to

$$\begin{aligned} \langle X | \mathbf{L}_\mu | X \rangle &= \sum_{jk} \sqrt{\lambda_j \lambda_k} \langle a_j | E_\mu^\dagger E_\mu | a_k \rangle \langle f_j | f_k \rangle \\ &= \sum_j \lambda_j \langle a_j | \mathbf{T}_\mu | a_j \rangle, \end{aligned} \quad (\text{B5})$$

because the set $\{|f_j\rangle\}$ is orthonormal. Due to Eq. (B5), the left-hand side of Eq. (B1) is equal to the left-hand side of Eq. (B3), and so we obtain Eq. (4.6).

Summing Eq. (B5) over all μ 's, we at once obtain

$$\langle X | \mathbf{L} | X \rangle = \sum_j \lambda_j \langle a_j | \mathbf{T} | a_j \rangle.$$

Therefore, the left-hand side of Eq. (B2) is equal to the left-hand side of Eq. (B4), and so we get Eq. (4.5).